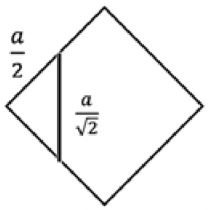
SOLUTION T1 [10 pts]. Three problems

T1-A [3.0 pts]. Hydrostatic gate

T1-A1 [3.0 pts].

Since the dam is perfectly sealed, the area of the opening is $a \cdot (a/\sqrt{2})$:



The horizontal force exerted by the water on either side is equal to the pressure at the centroid of the cube sections on the corresponding side times the effective area, as calculated previously:

$$\Delta F = \frac{a^2}{\sqrt{2}}\rho_0 g \Delta h$$

With an effective application point at the centroid of the left region of the block. The other relevant forces are the block's weight $\rho \times g \times vol$, the vertical component of the force exerted by the fluid (buoyant force) $\rho_0 \times g \times vol$, and the force that the hinge applies to the block (which exerts no torque).
At the critical Δh, the bottom edge of the opening applies zero force to the block. For the net torque to be 0,

$$\frac{a^2}{\sqrt{2}}\rho_0 g \Delta h \left(\frac{a}{2\sqrt{2}}\right) = (\rho - \rho_0)a^3 g \left(\frac{a}{2\sqrt{2}}\right)$$

Note: Arms are $a/(2\sqrt{2})$ from O either in the horizontal or vertical direction for pressure difference and effective weight respectively.

$$a = \frac{\rho_0}{\sqrt{2}(\rho - \rho_0)}\Delta h$$

For $\rho = 3\rho_0$,

$$\boxed{a = \frac{\Delta h}{2\sqrt{2}} = 0.50m}$$

T1-B [2.5 pts]. Electron-positron pair

From the definition of angular momentum,

$$mv_0(50a_0) = \mu\hbar$$

$$v_0 = \frac{\mu\hbar}{50ma_0}$$

The total energy of the system is

$$2\left(\frac{1}{2}mv_0^2\right) - \frac{ke^2}{100a_0} = \frac{ke^2}{a_0}\left(\frac{\mu^2}{50^2} - \frac{1}{100}\right)$$

T1-B1 [1.0 pts].

For $\mu = 4$,

$$E = -\frac{9ke^2}{2500a_0}$$

SOLUTION 1:
At the critical separation distances, $\vec{v} \perp \vec{r}$ and $L = mvr$. The energy at r_{max} and r_{min} can be written as

$$E = 2\frac{L^2}{2mr^2} - \frac{ke^2}{r}$$

Using $L = 2\mu\hbar = 8\hbar$ and the conservation of energy,

$$-\frac{9ke^2}{2500a_0} = \frac{64\hbar^2}{mr^2} - \frac{ke^2}{r}$$

$$\frac{ke^2}{a_0}\left(\frac{9}{2500}r^2 - a_0 r + 64a_0^2\right) = 0$$

$$r = \frac{a_0 \pm \sqrt{a_0^2 - \frac{576}{625}a_0^2}}{9/1250}$$

$$r = \frac{1250}{9}a_0\left(1 \pm \frac{7}{25}\right)$$

The maximum separation corresponds to the solution with the plus sign,

$$\boxed{r_{max} = \frac{1600}{9}a_0}$$

SOLUTION 2:
Since the energy is negative and the electrostatic force is central and proportional to r^{-2}, the trajectory of the particles will be elliptical with respect to the CM. The energy can be written in terms of the semi-major axis x as

$$E = -\frac{ke^2}{2x}$$

$$x = \frac{1250}{9}a_0$$

Since $\vec{v}_0$ is perpendicular to the vector that connects the two particles, the initial position must correspond to either r_{max} or r_{min}. The other critical distance, r_2 is given by the definition of x:

$$2x = r_0 + r_2$$

$$\boxed{r_2 = \left(\frac{2500}{9} - 100\right)a_0 = \frac{1600}{9}a_0}$$

Which is greater than r_0 and corresponds to r_{max}

SOLUTION 3:
The general equation for the trajectory in polar coordinates is

$$r(\theta) = \frac{\ell}{1+\varepsilon\cos(\theta-\theta_0)}$$

with $\ell = \frac{2L^2}{mke^2}$ the semi-latus rectum and ε the eccentricity, as given by the hint in question T1-B2. The maximum separation distance is reached when $\cos(\theta-\theta_0) = -1$:

$$r_{max} = \frac{\ell}{1-\varepsilon}$$

Calculating the values of ℓ and ε,

$$\ell = \frac{2(8\hbar)^2}{mke^2} = 128a_0$$

$$\varepsilon = \sqrt{1 - \frac{36ke^2(8\hbar)^2}{2500a_0 m(ke^2)^2}}$$

$$\varepsilon = \sqrt{1-\frac{2304}{2500}} = \frac{7}{25}$$

Replacing these values in the expression for r_{max},

$$\boxed{r_{max} = \frac{128}{1-(7/25)}a_0 = \frac{1600}{9}a_0}$$

T1-B2 [2.5 pts].
For $\mu = 15/2$,

$$E = +\frac{ke^2}{80a_0}$$

The distance between particles follows

$$r(\theta) = \frac{\ell}{1+\varepsilon\cos(\theta-\theta_0)}$$

Since the initial position corresponds to the closest approach, it is convenient to measure the angle θ with respect to the y axis so that $\cos\theta_0 = 1$, $\theta_0 = 0$. For $r \to \infty$,

$$1 + \varepsilon\cos\theta' \to 0$$

$$\theta' = \arccos(-1/\varepsilon)$$

Calculating the eccentricity,

$$\varepsilon = \sqrt{1 + \frac{4ke^2(15\hbar^2)}{80a_0 m(ke^2)^2}}$$

$$\varepsilon = \sqrt{1+\frac{45}{4}} = \frac{7}{2}$$

Replacing this value in the expression for θ,

$$\theta' = \arccos(-2/7) = 106.60^\circ$$

Since this is the angle with respect to the y-axis but the initial e^+ velocity is directed towards the positive x-axis, heading downward

$$\boxed{\theta = 90^\circ - \theta' = -16.60^\circ}$$

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T1-C [3.5 pts]. Photodissociation of ozone

T1-C1 [2.5 pts].

SOLUTION 1: pure algebra

The momentum conservation is

$$\hbar\frac{\omega}{c}\hat{k} = \vec{p} + \vec{q},$$

where $\vec{p}$ and $\vec{q}$ are the momentum of the O_2 and the O, respectively. Solving for q^2,

$$q^2 = \left(\frac{\hbar\omega}{c} - p\cos(\theta)\right)^2 + p^2\sin^2(\theta).$$

The energy conservation reads

$$\hbar\omega = \Delta U + \frac{p^2}{4m} + \frac{q^2}{2m}$$

where $\Delta U = U_f - U_i$.

For a given θ, p can be written in terms of ω. Replacing q^2 on energy balance and simplifying, we get

$$p^2 - \frac{4\hbar\omega\cos(\theta)}{3c}p + \frac{2\hbar^2\omega^2}{3c^2} + \frac{4}{3}m(\Delta U - \hbar\omega) = 0.$$

This gives a quadratic equation on p. For p to be real, the discriminant must be positive.

$$\frac{16\hbar^2\omega^2\cos^2(\theta)}{9c^2} - 4\left(\frac{2\hbar^2\omega^2}{3c^2} + \frac{4}{3}m(\Delta U - \hbar\omega)\right) \geq 0$$

The previous bound determines $\omega_{\min}$ as the minimal root of the quadratic equation on ω

$$(\cos(2\theta) - 2)\hbar^2\omega^2 + 6mc^2\hbar\omega - 6\Delta U mc^2 = 0$$

$$\omega_\pm = \frac{-6mc^2\hbar \pm \sqrt{D}}{2\left(\cos(2\theta) - 2\right)\hbar^2}$$

$$D = 36\hbar^2m^2c^4 + 24\Delta U mc^2\hbar^2\left(\cos(2\theta) - 2\right)$$

The smallest root ω_+. Thus, simplifying, one gets

$$\boxed{\omega_{\min} = \frac{3mc^2 - \sqrt{3mc^2\left(3mc^2 + 2\Delta U(\cos(2\theta) - 2)\right)}}{\hbar(2 - \cos(2\theta))}}$$

SOLUTION 2: Center of mass approach

Part I: Conservation laws

First, find the velocity of the center of mass. We get

$$(3mv_{CM}) = p_\gamma = \hbar\omega/c$$

We will use p_γ for calculations

$$v_{CM} = p_\gamma/(3m)$$

We want to relate the relative velocities in CM. We have by magnitude

$$2mv_1 = mv_2$$

And v_2 is the single oxygen atom (the one we don't want). The energy conservation equation is

$$\begin{aligned} E_\gamma = p_\gamma c &= \frac{1}{2}(3m)v_{CM}^2 + \Delta U + \frac{1}{2}mv_2^2 + \frac{1}{2}(2m)v_1^2 \\ &= \Delta U + \frac{p_\gamma^2}{6m} + \frac{1}{2}\times 6m(v_1)^2 \end{aligned}$$

We can now find v_1 in terms of p_γ.

Part 2: geometry and critical condition

We see that the set of possible velocities $\vec{v}_1$ form a circle (they have constant v_1). We need to consider that θ is the angle between $\vec{v}_{CM}$ and $\vec{v}_{CM} + \vec{v}_1$ (this is the full velocity of particle 1).

If the circle of radius v_1 is too small there is no solution (the straight line at the angle will not intersect the circle). If the radius v_1 is big enough, there will always be two solutions intersecting the circle. The critical case (crossover from no solution to two) is when the circle is tangent to the straight line. This is the case we need.

In that case the triangle formed by $0, \vec{v}_{CM}, \vec{v}_{CM} + \vec{v}_1$ is a right triangle.

Warning: This can only happen if $\theta \leq 90^o$. The other case will be dealt with separately.

This way

$$v_1 = v_{CM}\sin(\theta)$$

Replace above to find that in conservation of energy

$$p_\gamma c = \Delta U + \frac{p_\gamma^2}{6m} + \frac{1}{2}\times\frac{p_\gamma^2\sin^2(\theta)}{3m}$$

now we can solve the quadratic equation for p_γ. We get (*we need to choose the correct solution of the quadratic equation*)

$$p_\gamma = \frac{3cm - \sqrt{3}\sqrt{m}\sqrt{3c^2m + 2\Delta U\cos(2\theta) - 4\Delta U}}{2\sin^2(\theta)+1}$$

Note: Any of the following expressions are the same:

$$2\sin^2(\theta)+1 = 3 - 2\cos^2(\theta) = 2 - \cos(2\theta)$$

Rewrite as the angular frequency

$$\boxed{\omega_{min} = 3mc^2\left[\frac{1-\sqrt{1-\frac{\Delta U}{3mc^2}(2\sin^2(\theta)+1)}}{(2\sin^2(\theta)+1)\hbar}\right]}$$

Special case: $\theta \geq 90^0$, *this is essentially impossible to see in a brute force algebraic solution, but it is an important feature of the problem. Needs a small amount of points in this solution*

For angle greater than 90^0, as soon as the circle reaches the answer for 90^0 it becomes big enough to allow all angles. Formula is the same with $\theta = 90^0$.

T1-C2 [1.0 pts].

Up to second order in ΔU

$$\omega_{\min} = \frac{\Delta U}{\hbar} - \frac{(\cos(2\theta)-2)\Delta U^2}{6\hbar mc^2}$$

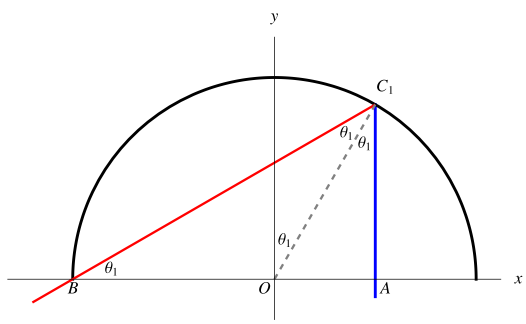
$$\hbar\omega_{\min} - \Delta U = \frac{(2-\cos(2\theta))}{6mc^2}(\Delta U)^2$$

$$\boxed{\hbar\omega_{\min} - \Delta U = 2.03\times 10^{-11}\text{eV}}$$

SOLUTION T2 [10 pts]. Cooking with sunlight?

PART A [1.5 pts]. Multiple reflections

T2-A1 [1.5 pts]. For $n = 1$, finding x_1 corresponds to considering the limiting case of the ray, shown in blue in the figure below.



In this case, the incident ray passes through point A, and the reflected ray, shown in red in the figure, passes exactly through point B on the edge of the mirror.
Let θ_1 be the angle formed by the radius passing through the reflection point C_1 with the vertical axis (the y axis). Let O be the center of the mirror.
Since the incident ray is parallel to the y axis, the angle of incidence is

$$\angle OC_1A = \theta_1\,.$$

By the law of reflection,

$$\angle OC_1A = \angle OC_1B = \theta_1\,.$$

Additionally, note that the triangle C_1BO is isosceles, so

$$\angle C_1BO = \theta_1\,.$$

Finally, summing the internal angles of the triangle C_1BA, we have

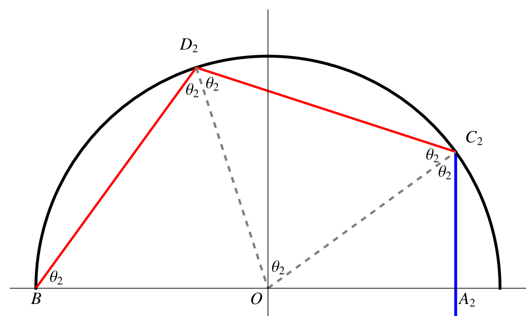
$$3\theta_1 + \frac{\pi}{2} = \pi,$$

$$\theta_1 = \frac{\pi}{6}.$$

Correspondingly, from the triangle OC_1A, we obtain

$$\boxed{x_1 = R\sin\frac{\pi}{6} = \frac{R}{2}.}$$

For $n = 2$, the limiting case for two reflections is shown in the figure below.



In analogy with the case $n = 1$, we let θ_2 be the angle between the radius passing through the reflection point C_2 and the mirror axis. We see that the trajectory of the reflected ray determines two isosceles triangles with a common side and a base angle of θ_2. Also note that the ray's trajectory forms half of a regular pentagon. Summing over the internal angles of the two isosceles triangles and the rectangular triangle, we obtain

$$5\theta_2 + \frac{\pi}{2} + \pi = 3\pi,$$

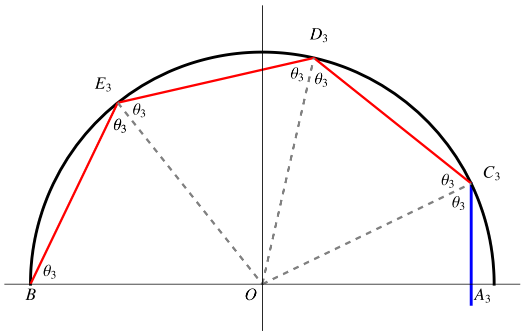
which simplifies to

$$\theta_2 = \frac{3\pi}{10}.$$

Correspondingly,

$$\boxed{x_2 = R\sin\frac{3\pi}{10} = \frac{1+\sqrt{5}}{4}R \approx 0.8R.}$$

Case n=3: For completeness, we present the limiting case for 3 reflections in the figure below.



Letting θ_3 be the incidence angle, we observe that the trajectory of the reflected ray determines three isosceles triangles with common sides and a base angle of θ_3. Also notice that the ray's trajectory now forms half of a regular heptagon. Summing over the internal angles of the three isosceles triangles and the rectangular triangle, we obtain

$$7\theta_3 + \frac{\pi}{2} + \pi = 4\pi,$$

which simplifies to

$$\theta_3 = \frac{5\pi}{14}$$

Correspondingly

$$\boxed{x_3 = R\sin\frac{5\pi}{14} \approx 0.9R.}$$

Generalizing the result, the limiting case for n reflections corresponds to the ray with incidence angle θ_N.

The trajectory of the reflected ray determines n isosceles triangles with common sides and a base angle of θ_N.

Also note that the ray's trajectory now forms half of a regular $2n+1$-gon. Summing over the internal angles of the n isosceles triangles and the rectangular triangle determined by the incident ray, we obtain

$$(2n\theta_N + \pi) + \left(\frac{\pi}{2} + \theta_N\right) = (n+1)\pi,$$

$$(2n+1)\theta_N + \frac{3\pi}{2} = (n+1)\pi,$$

which simplifies to

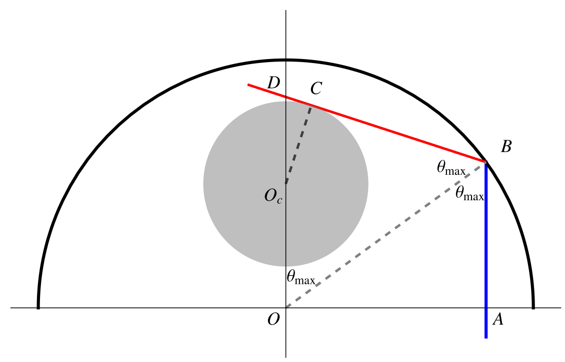
$$\theta_N = \frac{2n-1}{4n+2}\pi$$

Correspondingly

$$\boxed{x_n = R\sin\frac{2N-1}{4N+2}\pi = R\cos\frac{\pi}{2N+1}}$$

T2-B [4.0 pts]. Solar Cooker

T2-B1 [2.0 pts].
The limiting case is shown in the figure below.



- O_c is the center of the container.
- A is the incidence point on the x-axis
- B is the reflection point
- The reflected ray is tangential to the container at the point C
- The reflected ray crosses the y-axis at D.

As the triangle ODB is isosceles,

$$\angle ODB = \pi - 2\theta_{max}\,,$$

$$\cos\theta_{max} = \frac{R}{2\overline{OD}}.$$

$$\overline{OD} = \frac{R}{2\cos\theta_{max}}.$$

The triangle O_cDC is rectangle

$$\begin{aligned}
a &= \overline{O_cD}\sin(\pi - 2\theta_{\max}) \\
&= \left(\overline{OD} - \frac{R}{2}\right)\sin(2\theta_{\max}) \\
&= \left(\frac{R}{2\cos\theta_{\max}} - \frac{R}{2}\right)\sin(2\theta_{\max}) \\
&= R\sin\theta_{\max} - \frac{R}{2}\sin(2\theta_{\max})
\end{aligned}$$

The coefficients read

$$\boxed{\alpha = R}$$

$$\boxed{\beta = -\frac{R}{2}}$$

T2-B2 [1.5 pts]. Without the mirror:

$$P_0 = 2Ial,$$

where I is the sunlight intensity and l is the length of the cylinder.
With the mirror:

$$\begin{aligned}
P &= 2I\overline{OA}l \\
&= 2IRl\sin\theta_{\max}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{P}{P_0} &= \frac{2IRl\sin\theta_{\max}}{2Ial} \\
&= \frac{R\sin\theta_{\max}}{R\sin\theta_{\max}(1-\cos\theta_{\max})}
\end{aligned}$$

$$\boxed{\frac{P}{P_0} = \frac{1}{1-\cos\theta_{\max}}.}$$

Alternative answer in terms of α and β: Without the mirror:

$$P_0 = 2Ial = 2I(\alpha\sin\theta_{\max} + \beta\sin(2\theta_{\max}))l$$

where I is the sunlight intensity and l is the length of the cylinder.

With the mirror:

$$\begin{aligned}
P &= 2I\overline{OA}l \\
&= 2IRl\sin\theta_{\max}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{P}{P_0} &= \frac{2IRl\sin\theta_{\max}}{2Ial} \\
&= \frac{R\sin\theta_{\max}}{\alpha\sin\theta_{\max} + \beta\sin(2\theta_{\max})}
\end{aligned}$$

$$\boxed{\frac{P}{P_0} = \frac{R}{\alpha + 2\beta\cos\theta_{\max}}.}$$

T2-B3 [0.5 pts]. If $P = 5P_0$

$$\begin{aligned}
5 &= \frac{1}{1-\cos\theta_{\max}}, \\
\cos\theta_{\max} &= \frac{4}{5}, \\
\sin\theta_{\max} &= \frac{3}{5}.
\end{aligned}$$

Finally

$$a = (1\text{m})\frac{3}{5}\left(1 - \frac{4}{5}\right).$$

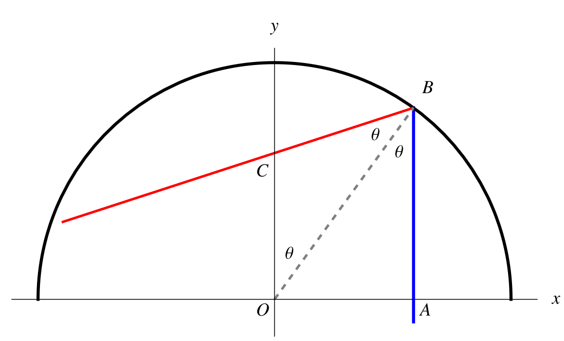
$$\boxed{a = 12\,\text{cm}.}$$

Alternative solution parametrized in terms of α and β

$$a = -\frac{R\sqrt{100\beta^2 - (R-5\alpha)^2}}{50\beta}$$

T2-C [4.5 pts]. Caustics and Cusp

T2-C1 [0.5 pts].



The reflected ray forms an angle 2θ with the vertical axis. Therefore, the slope of the line is

$$\boxed{m_{_A} = \cot(2\theta).}$$

The reflected ray intersects the y-axis at C, $\overline{OC} = b_A$, forming the isosceles triangle OBC. from the intersection

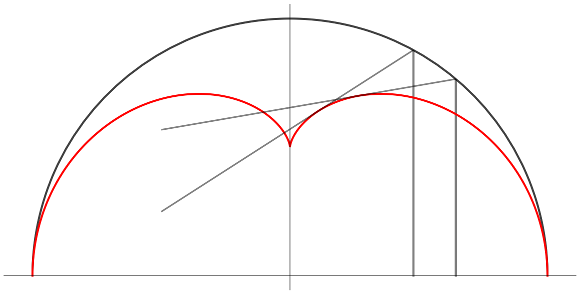
$$\cos\theta = \frac{R}{2b_A}.$$

$$\boxed{b_A = \frac{R}{2\cos\theta}.}$$

Equivalent expressions:

$$m_{_A} = \cot(2\theta) = \frac{1}{\tan(2\theta)} = \frac{\cot(\theta)}{2} - \frac{\tan(\theta)}{2}$$
$$b_A = \frac{R}{2\cos\theta} = \frac{R}{2}\sec\theta$$

T2-C2 [2.0 pts].



The slope is

$$\begin{aligned}
m_{_B} &= \cot(2\theta + 2\Delta\theta) \\
&= \frac{\cot(2\theta)\cot(2\Delta\theta) - 1}{\cot(2\theta) + \cot(2\Delta\theta)} \\
&= \frac{\cot(2\theta) - \tan(2\Delta\theta)}{\cot(2\theta)\tan(2\Delta\theta) + 1} \\
&= (\cot(2\theta) - 2\Delta\theta)(1 - 2\Delta\theta\cot(2\theta)) + \dots \\
&= \cot(2\theta) - 2\Delta\theta(1 - \tan^2(2\Delta\theta)) + \dots
\end{aligned}$$

Then, the first-order expansion of $m_{_B}$ reads

$$\boxed{m_{_B} = \cot(2\theta) - 2\csc^2(2\theta)\Delta\theta.}$$

On the other hand, the intercept is

$$\begin{aligned}
b_{_B} &= \frac{R}{2}\sec(\theta + \Delta\theta) \\
&= \frac{R}{2}\frac{\sec(\theta)\sec(\Delta\theta)}{1 - \tan(\theta)\tan(\Delta\theta)} \\
&= \frac{R}{2}\left(\sec(\theta)(1 + \tan(\theta)\Delta\theta)\right) + \dots
\end{aligned}$$

and the first-order expansion reads

$$\boxed{b_{_B} = \frac{R}{2}\sec(\theta)\left(1 + \tan(\theta)\Delta\theta\right)}$$

Equivalent expressions:

$$\begin{aligned}
m_{_B} &= \cot(2\theta) - 2\csc^2(2\theta)\Delta\theta \\
&= \frac{1}{2}\csc^2(2\theta)(\sin(4\theta) - 4\Delta\theta) \\
&= \frac{1}{2}\left(\cot(\theta) - \tan(\theta) - \csc^2(\theta)\sec^2(\theta)\right)\Delta\theta
\end{aligned}$$

$$\begin{aligned}
b_{_B} &= \frac{R}{2}\sec(\theta)\left(1 + \tan(\theta)\Delta\theta\right) \\
&= \frac{R}{2\cos(\theta)}\left(1 + \tan(\theta)\Delta\theta\right)
\end{aligned}$$

T2-C3 [1.0 pts].
The coordinates of the intersection point, (X_C, Y_C), fulfill the following equations:

$$\begin{aligned}
Y_C &= m_{_A} X_C + b_{_A} \\
Y_C &= m_{_B} X_C + b_{_B},
\end{aligned}$$

Solving:

$$(X_C, Y_C) = \left(-\frac{b_{_A} - b_{_B}}{m_{_A} - m_{_B}}, \frac{m_{_A} b_{_B} - b_{_A} m_{_B}}{m_{_A} - m_{_B}}\right)$$

Replacing $m_{_B}$ and $b_{_B}$, we find:

$$(X_C, Y_C) = \left(R\sin^3(\theta), \frac{R}{2}\cos(\theta)(2 - \cos(2\theta))\right)$$

$$\boxed{X_C = R\sin^3(\theta)}$$

$$\boxed{Y_C = \frac{R}{2}\cos(\theta)(2-\cos(2\theta))}$$

Equivalent expressions:

$$\begin{aligned} Y_C &= \frac{R}{2}\cos(\theta)(2-\cos(2\theta)) \\ &= \frac{R}{4}(3\cos(\theta)-\cos(3\theta)) \\ &= \frac{R}{4}(3\cos(\theta)-\cos^3(\theta)+3\sin^2(\theta)\cos(\theta)) \end{aligned}$$

T2-C4 [1.0 pts]. For $\theta \ll 1$, we use the approximations

$$\sin\theta \approx \theta,$$
$$\cos\theta \approx 1-\frac{\theta^2}{2}.$$

Thus, near the cusp, the caustic is approximated as

$$(X_C, Y_C) \approx \left(R\theta^3, \frac{R}{4}(2+3\theta^2)\right)$$

$$(X_C, Y_C) \approx \left(R\theta^3, \frac{R}{2}+\frac{3R}{4}\theta^2\right)$$

Solving for θ, we get

$$\theta = \left(\frac{X_C}{R}\right)^{\frac{1}{3}},$$

and substitute in Y_C

$$Y_C = \frac{R}{2}+\frac{3R}{4}\left(\frac{X_C}{R}\right)^{\frac{2}{3}}$$

Comparing with

$$Y_C = v|X_C|^{p/q}+u,$$

we find

$$\boxed{v = \frac{3}{4}R^{\frac{1}{3}}}$$

$$\boxed{u = \frac{R}{2},}$$

$$\boxed{p=2}$$

$$\boxed{q=3}$$

This shows that the shape of the cusp is universal and that its tip is located at $y = \frac{R}{2}$.

SOLUTION T3 [10 pts]. Chashing abolute zero

T3-A [1.0 points]

T1-A1 [0.2 pts]. Field $\vec{H}$ inside the torus is almost entirely confined to its core, forming concentric circumferential circles. Therefore, from magnetic circuital law $\oint_C \vec{H}\cdot d\vec{\ell} = I_C$ we have $H\cdot 2\pi R = NI$. Dividing and multiplying by $A=\pi r^2$ yields to

$$H = \frac{NI}{2\pi R}\frac{A}{\pi r^2} = \frac{NIA}{V}.$$

T3-A2 [0.6 pts]. When I is changed, there is induced an *emf* $\mathscr{E}_{\text{ind}}$. Thus, since

$$\mathscr{E} + \mathscr{E}_{\text{ind}} = \text{resistance}\times I \approx 0,$$

from Faraday's law of induction, the work done by the external *emf* is given by

$$dW_{emf} = \mathscr{E}dq = -\mathscr{E}_{\text{ind}}Idt = \frac{d\Phi_B}{dt}Idt = Id\Phi_B,$$

where dq is the quantity of electric charge flowing during the time interval dt and $\Phi_B = NAB$ is the flux of the field $\vec{B}$ inside the torus. Therefore,

$$dW_{emf} = NIA\ dB = VH\ dB.$$

T3-A3 [0.2 pts]. Since $\vec{B}$, $\vec{H}$ and $\vec{M}$ are paralell vectors, from $\vec{B} = \mu_0\vec{H} + \mu_0\vec{M}$ we get

$$dB = \mu_0 dH + \mu_0 dM,$$

from which, after substitution in the answer of question T3-A2, yields to

$$dW_{emf} = \mu_0 VHdH + \mu_0 VHdM.$$

Hence,

$$dW = \mu_0 VHdM.$$

T3-B [3.0 pts]

T3-B1 [1.5 pts]. The first law of thermodynamics for the torus is $dU = dW + dQ = \mu_0 VHdM + CdT$. At constant M, we have $dU = C_M dT = n\lambda dT/T^2$, i. e., U depends only on T. Hence, for isothermal processes, $dQ = -dW = -\mu_0 VHdM$. Furthermore, from the equation of state we have for isothermal processes that $TVdM = nKdH$ and then

$$dQ = -\mu_0 VH\left(\frac{nK}{TV}dH\right) = -\mu_0\frac{nK}{T}HdH.$$

Integration from H_i to H_f gives

$$Q = -\mu_0\frac{nK}{T}\frac{H_f^2 - H_i^2}{2}.$$

Note: This is proportional to the volume because it has the number of moles.

T3-B2 [1.5 pts]. For adiabatic processes, the first law yields to $dU = dW$ and then

$$\frac{n\lambda}{T^2}dT = \mu_0 VHdM = \mu_0 V\frac{TMV}{nK}dM,$$

where we have used the equation of state to substitute H. Separating variables we have

$$\frac{n\lambda}{T^3}dT = \mu_0\frac{V^2}{nK}MdM,$$

and hence, after integration from (T_i, M_i) to (T_f, M_f),

$$\frac{n\lambda}{2}\left(\frac{1}{T_i^2} - \frac{1}{T_f^2}\right) = \mu_0\frac{V^2}{nK}\frac{M_f^2 - M_i^2}{2}.$$

Since $M = nKH/TV$, this equation simplifies to

$$\frac{\lambda}{T_i^2} - \frac{\lambda}{T_f^2} = \mu_0 K\left(\frac{H_f^2}{T_f^2} - \frac{H_i^2}{T_i^2}\right),$$

from which

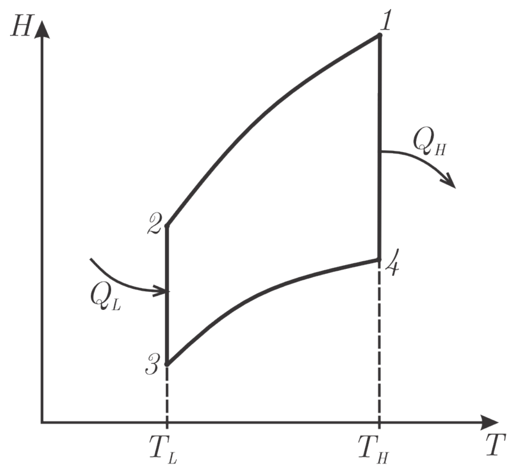
$$\frac{T_f^2}{T_i^2} = \frac{\lambda + \mu_0 KH_f^2}{\lambda + \mu_0 KH_i^2}.$$

Hence

$$\Delta T = T_f - T_i = T_i\left(\sqrt{\frac{\lambda + \mu_0 KH_f^2}{\lambda + \mu_0 KH_i^2}} - 1\right).$$

T3-C [6.0 pts]

T3-C1 [0.2 pts].



T3-C2 [1.5 pts]. In the Carnot cycle

$$\frac{Q_c}{Q_h} = \frac{T_c}{T_h}.$$

Particularly, for the Carnot cycle of paramagnetic refrigeration, we have $Q_c = |Q_{2\to3}| = Q_{2\to3}$ and $Q_h = |Q_{4\to1}| = -Q_{4\to1}$. From the answer to question T3-B1,

$$Q_c = -\mu_0 \frac{nK}{T_c}\frac{H_3^2 - H_2^2}{2} = \mu_0 \frac{nK}{T_c}\frac{H_2^2 - H_3^2}{2},$$
$$Q_h = -\left(-\mu_0 \frac{nK}{T_h}\frac{H_1^2 - H_4^2}{2}\right) = \mu_0 \frac{nK}{T_h}\frac{H_1^2 - H_4^2}{2},$$

where $T_c = T_2 = T_3$ and $T_h = T_4 = T_1$ are the respective temperatures of the isothermal processes $2 \to 3$ and $4 \to 1$. Therefore, after substitution and simplification in the Carnot identity, we have

$$\frac{T_h}{T_c}\frac{H_2^2 - H_3^2}{H_1^2 - H_4^2} = \frac{T_c}{T_h}, \quad \frac{H_2^2 - H_3^2}{T_c^2} = \frac{H_1^2 - H_4^2}{T_h^2}.$$

Since, by the equation of state, $M = nKH/TV$, we get

$$M_2^2 - M_3^2 = M_1^2 - M_4^2, \; M_1 = \sqrt{M_2^2 - M_3^2 + M_4^2}.$$

T3-C3 [0.8 pts]. By substituting the given numerical values

$$Q_c = \mu_0 \frac{nK}{T_c}\frac{H_2^2 - H_3^2}{2} = 1.29 \times 10^{-1} \text{ J}.$$

Hence

$$|\Delta T|_{\textbf{helium}} = \frac{1.29 \times 10^{-1} \text{ J}}{130 \text{ kg} \cdot \text{m}^{-3} \times 10^{-3} \text{ m}^3 \times 10^2 \text{ J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}}$$
$$= \frac{1.29}{1.30} \times 10^{-2} \text{ K} = 9.92 \times 10^{-3} \text{ K}.$$

Hence

$$T_f = T_i - |\Delta T| = 1.00 \text{ K} - 0.00992 \text{ K} = 0.99008 \text{ K}.$$

T3-C4 [2.0 pts]. From the Carnot cycle identity $dQ_c/dQ_h = T_c/T_h$ and the first law in differential form $dQ_h = dW + dQ_c$, we have

$$\frac{dW + dQ_c}{dQ_c} = \frac{dW}{dQ_c} + 1 = -\frac{Pdt}{C_c dT_c} + 1 = \frac{T_h}{T_c},$$

where $dW = Pdt$ and $dQ_c = -C_c dT_c$ have been used. Hence, after separation of variables,

$$dt = -\frac{C_c T_h}{P}\left(\frac{dT_c}{T_c} - \frac{dT_c}{T_h}\right).$$

Integrating from $t = 0$ ($T_c = T_0$) to t ($T_c = T$) we get

$$t = \frac{C_c T_h}{P}\left(\ln\frac{T_0}{T} - \frac{T_0 - T}{T_h}\right).$$

T3-C5 [1.5 pts]. Since $W = Pt$ and $Q_c = C_c(T_0 - T)$,

$$W = Pt = C_c T_h \ln\frac{T_0}{T} - Q_c.$$

Therefore

$$\text{COP} = \frac{Q_c}{W} = \left(\frac{T_h}{T_0 - T}\ln\frac{T_0}{T} - 1\right)^{-1}.$$